

Conley index for random dynamical systems*

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Abstract

Conley index theory is a very powerful tool in the study of dynamical systems, differential equations and bifurcation theory. In this paper, we make an attempt to generalize the Conley index to discrete random dynamical systems. And we mainly follow the Conley index for maps given by Franks and Richeson in [6]. Furthermore, we simply discuss the relations of isolated invariant sets between time-continuous random dynamical systems and the corresponding time- h maps. For applications we give several examples to illustrate our results.

Key words: Conley index; Random dynamical systems

1 Introduction

One important aspect of the qualitative analysis of differential equations and dynamical systems is the study of asymptotic, long-term behavior of solutions/orbits. Hence much of dynamical systems involves the study of the existence and structure of invariant sets. The Conley index theory (see [3, 4, 18]), developed by Conley and his students, has been a very powerful tool in the study of dynamical systems, differential equations and bifurcation theory. The Conley index theory was developed for flows on compact spaces at first, then Rybakowski [17] extended the theory to semiflows on noncompact spaces. After that, a natural question is to find an appropriate generalization to discrete dynamical systems (maps). Robbin and Salamon [15] presented the first version of the Conley index for maps, which was defined in terms of shape theory. Later Mrozek [12] offered a cohomological definition of Conley index for maps based on the Leray reduction. Szymczak [20] constructed the homotopy Conley index as a functor into an abstract category and showed that any other Conley index can be factorized through his homotopy Conley index. Recently, a more intuitive definition of the discrete Conley index was given by Franks and Richeson

*This work is partially supported by ‘985’ project of Jilin University and graduate innovation lab of Jilin University.

[6], which gives an accessible and intuitive development of a Conley index for isolated invariant sets of any continuous map defined on a locally compact metric space. They also showed that their definition of the discrete Conley index based on shift equivalences is identical to Szymczak's categorical definition of the Conley index.

Random dynamical systems arise in the modeling of many phenomena in physics, biology, economics, climatology, etc and the random effects often reflect intrinsic properties of these phenomena rather than just to compensate for the defects in deterministic models. The history of study of random dynamical systems goes back to Ulam and von Neumann [21] and it has flourished since the 1980s due to the discovery that the solutions of stochastic ordinary differential equations yield a cocycle over a metric dynamical system which models randomness, i.e. a random dynamical system. In this paper we make an attempt to obtain the Conley index for random dynamical systems. The definition of Conley index given by Franks and Richeson [6] is relative simple and unlike index pairs, the filtration pairs they defined are robust under small C^0 perturbations of the map f , which is a wonderful property to obtain the continuation property of Conley index immediately. Hence we will follow [6] to obtain the Conley index for discrete random dynamical systems. Such Conley index is very useful in studying random homeomorphisms, discretized random differential equations etc, see Section 8 for several simple applications. Furthermore we also discuss the isolated invariant set for time-continuous random dynamical systems and its relation with the isolated invariant set for the time- h maps. This indicates that we can obtain some information of time-continuous random dynamical system by studying its time- h maps.

The paper is organized as follows. In Section 2, we recall some basic definitions and results for random dynamical systems. In Section 3, we give the definitions of random isolating neighborhood and random isolated invariant set and compare the random isolated invariant set and random omega-limit set. In Section 4, we prove that for any random neighborhood of a random isolated invariant set, there is a random filtration pair for it in this neighborhood. In Section 5, we introduce the random shift equivalence and show that for any two random filtration pairs of a random isolated invariant set, the corresponding two random pointed spaces with their random pointed space maps are random shift equivalent. In Section 6 we give the definition of random Conley index for random isolated invariant sets. In Section 7 we simply discuss the relation of isolated invariant sets between time-continuous random dynamical system and the discrete one generated by its time- h maps. And at last for applications we give several simple examples to illustrate our results in Section 8.

2 Random dynamical systems

In this section, we will give some preliminary definitions and propositions for the later use. Firstly we give the definition of continuous random dynamical systems (cf Arnold [1]).

Definition 2.1. Let X be a metric space with a metric d_X . A continuous *random dynamical system (RDS)*, shortly denoted by ϕ , consists of two ingredients:

- (i) A model of the noise, namely a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(t, \omega) \mapsto \theta_t \omega$ is a measurable flow which leaves \mathbb{P} invariant, i.e. $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{T}$.
- (ii) A model of the system perturbed by noise, namely a cocycle ϕ over θ , i.e. a measurable mapping $\phi : \mathbb{T} \times \Omega \times X \rightarrow X, (t, \omega, x) \mapsto \phi(t, \omega, x)$, such that $(t, x) \mapsto \phi(t, \omega, x)$ is continuous for all $\omega \in \Omega$ and the family $\phi(t, \omega, \cdot) = \phi(t, \omega) : X \rightarrow X$ of random self-mappings of X satisfies the cocycle property:

$$\phi(0, \omega) = \text{id}_X, \phi(t + s, \omega) = \phi(t, \theta_s \omega) \circ \phi(s, \omega) \quad \text{for all } t, s \in \mathbb{T}, \omega \in \Omega. \quad (1)$$

In this definition, $\mathbb{T} = \mathbb{Z}$ or \mathbb{R} .

In this paper, we mainly consider discrete random dynamical systems, i.e. $\mathbb{T} = \mathbb{Z}$. We will use $\theta := \theta_1$ to denote the time one map of θ_n . It follows from (1) that $\phi(n, \omega)$ is a homeomorphism of X , and the fact

$$\phi^{-1}(n, \omega) = \phi(-n, \theta_n \omega)$$

is very useful in the following.

Assume ϕ is a discrete random dynamical system and φ is the time-one map of ϕ , i.e. $\varphi(\omega) = \phi(1, \omega) : X \rightarrow X$, then we call φ the *random homeomorphism* determined by ϕ . On the other hand, assume φ is a random homeomorphism, then it generates a discrete RDS $\phi(n, \omega, x)$ in the following way:

$$\phi(n, \omega) = \begin{cases} \varphi(\theta_n \omega) \circ \cdots \circ \varphi(\omega), & n > 0, \\ \text{Id}_X, & n = 0, \\ \varphi^{-1}(\theta_n \omega) \circ \cdots \circ \varphi^{-1}(\theta_{-1} \omega), & n < 0. \end{cases}$$

Hence we will identify a random homeomorphism with the discrete RDS generated by it, which will not cause confusions. For a random homeomorphism φ , we will use φ^k to denote the k -times iteration of φ , i.e. $\varphi^k(\omega, \cdot) := \phi(k, \omega, \cdot)$, where ϕ is the discrete RDS generated by φ .

Below any mapping from Ω into the collection of all subsets of X is said to be a *multifunction* (or a set valued mapping) from Ω into X . We now give the definition of random set, which is a fundamental concept for RDS.

Definition 2.2. Let X be a metric space with a metric d_X . The multifunction $\omega \mapsto D(\omega)$ taking values in the closed/compact subsets of X is said to be a *random closed/compact set* if the mapping $\omega \mapsto \text{dist}_X(x, D(\omega))$ is measurable for any $x \in X$, where $\text{dist}_X(x, B) := \inf_{y \in B} d_X(x, y)$. The multifunction $\omega \mapsto U(\omega)$ taking values in the open subsets of X is said to be a *random open set* if $\omega \mapsto U^c(\omega)$ is a random closed set, where U^c denotes the complement of U .

Definition 2.3. A random set $D(\omega)$ is said to be *forward invariant* under the RDS ϕ if $\phi(t, \omega)D(\omega) \subset D(\theta_t \omega)$ for all $t \in \mathbb{T}_+$; It is said to be *backward invariant* if $\phi(t, \omega)D(\omega) \supset D(\theta_t \omega)$ for all $t \in \mathbb{T}_+$; It is said to be *invariant* if $\phi(t, \omega)D(\omega) = D(\theta_t \omega)$ for all $t \in \mathbb{T}$.

Throughout the paper, we will assume that (X, d_X) is a locally compact Polish space, i.e. a separable complete metric space. For a random variable $T(\omega)$, we call $T(\omega) > 0$ if it holds almost surely. We also call a multifunction $D(\omega)$ measurable for convenience if the mapping $\omega \mapsto \text{dist}_X(x, D(\omega))$ is measurable for any $x \in X$.

Now we enumerate some basic results about random sets in the following propositions, for details the reader can refer to Castaing and Valadier [2] and Arnold [1] for instance.

Proposition 2.1. *Let X be a Polish space, then the following assertions hold:*

- (i) *if D is a random closed set, then so is the closure of D^c ;*
- (ii) *if D is a random open set, then the closure $\text{cl}D$ of D is a random closed set;*
- (iii) *if D is a random closed set, then $\text{int}D$, the interior of D , is a random open set;*
- (iv) *if $\{D_n, n \in \mathbb{N}\}$ is a sequence of random closed sets and there exists $n_0 \in \mathbb{N}$ such that D_{n_0} is a random compact set, then $\bigcap_{n \in \mathbb{N}} D_n$ is a random compact set;*
- (v) *if $f : \Omega \times X \rightarrow X$ is a function such that $f(\omega, \cdot)$ is continuous for all ω and $f(\cdot, x)$ is measurable for all x , then $\omega \mapsto f(\omega, D(\omega))$ is a random compact set provided that $D(\omega)$ is a random compact set.*

The following measurable selection theorem is frequently used for our purpose, although we do not always mention it when we do use it. For the proof, we refer to [1, 2] for details.

Proposition 2.2. (Measurable Selection Theorem). *Let a multifunction $\omega \mapsto D(\omega)$ take values in the subspace of closed non-void subsets of a Polish space X . Then $D(\omega)$ is a random closed set if and only if there exists a sequence $\{v_n : n \in \mathbb{N}\}$ of measurable maps $v_n : \Omega \rightarrow X$ such that*

$$v_n(\omega) \in D(\omega) \quad \text{and} \quad D(\omega) = \text{cl}\left(\bigcup \{v_n(\omega), n \in \mathbb{N}\}\right) \quad \text{for all } \omega \in \Omega.$$

In particular if $D(\omega)$ is a random closed set, then there exists a measurable selection, i.e. a measurable map $v : \Omega \rightarrow X$ such that $v(\omega) \in D(\omega)$ for all $\omega \in \Omega$.

The following proposition comes from [16] and will be used later, which gives a relation between an $\tilde{\mathcal{F}}^\nu$ -measurable function and an \mathcal{F} -measurable one.

Proposition 2.3. *Assume ν is a positive measure on the measurable space (X, \mathcal{F}) . Denote $\tilde{\mathcal{F}}^\nu$ the completion of the σ -algebra \mathcal{F} with respect to the measure ν . If f is an $\tilde{\mathcal{F}}^\nu$ -measurable function, then there exists an \mathcal{F} -measurable function g such that $f = g$ a.e. $[\nu]$.*

3 Random isolated invariant sets and omega-limit sets

Definition 3.1. For given two random sets $D(\omega), A(\omega)$, we say $A(\omega)$ *attracts* $D(\omega)$ if

$$\lim_{n \rightarrow \infty} d(\phi(n, \theta_{-n}\omega)D(\theta_{-n}\omega)|A(\omega)) = 0$$

holds almost surely, where $d(A|B)$ stands for the Hausdorff semi-metric between two sets A, B , i.e. $d(A|B) := \sup_{x \in A} \inf_{y \in B} d(x, y)$.

For any given random set $D(\omega)$, we denote $\Omega_D(\omega)$ the *omega-limit set* of $D(\omega)$, which is defined as follows:

$$\Omega_D(\omega) := \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} \phi(k, \theta_{-k}\omega)D(\theta_{-k}\omega)}.$$

It is well-known that if a nonvoid random set $D(\omega)$ is attracted by a random compact set $A(\omega)$, then $\Omega_D(\omega) \neq \emptyset$ almost surely and it is invariant. Also it is known that for two random sets $D_1(\omega), D_2(\omega)$, if they are attracted by a random compact set $A(\omega)$, then we have

$$\Omega_{D_1 \cup D_2}(\omega) = \Omega_{D_1}(\omega) \cup \Omega_{D_2}(\omega).$$

This important fact will be used later, e.g. in Lemma 5.2. (In fact, this conclusion also holds if $D_1(\omega), D_2(\omega)$ are not attracted by a random compact set.)

Definition 3.2. A random compact set $N(\omega)$ is called a *random isolating neighborhood* if it satisfies

$$\text{Inv}N(\omega) \subset \text{int}N(\omega),$$

where $\text{int}N(\omega)$ denotes the interior of $N(\omega)$ and

$$\text{Inv}N(\omega) = \{x \in N(\omega) \mid \phi(n, \omega, x) \in N(\theta_n\omega), \forall n \in \mathbb{Z}\}.$$

Correspondingly we call $S(\omega)$ a *random isolated invariant set* if there exists a random isolating neighborhood $N(\omega)$ such that $S(\omega) = \text{Inv}N(\omega)$. A random compact set $N(\omega)$ is called a *random isolating block* if it satisfies

$$\varphi(\theta_{-1}\omega, N(\theta_{-1}\omega)) \cap N(\omega) \cap \varphi^{-1}(\theta\omega, N(\theta\omega)) \subset \text{int}N(\omega).$$

Definition 3.3. For a random set $N(\omega)$ we define the *exit set* of $N(\omega)$ to be

$$N^-(\omega) := \{x \in N(\omega) \mid \varphi(\omega, x) \notin \text{int}N(\theta\omega)\}.$$

Remark 3.1. It is obvious that a random isolating block is a random isolating neighborhood, but the converse is not true. It is easy to see that for a random isolating neighborhood $N(\omega)$, its corresponding random isolated invariant set $S(\omega)$ can be characterized by

$$S(\omega) = \text{Inv}N(\omega) = \bigcap_{n \in \mathbb{Z}} \phi(-n, \theta_n\omega)N(\theta_n\omega)$$

and the exit set $N^-(\omega)$ of a random set $N(\omega)$ can be characterized by

$$N^-(\omega) = N(\omega) \cap [\varphi^{-1}(\theta\omega, \text{int}^c N(\theta\omega))], \quad (2)$$

where $\text{int}^c N(\omega)$ denotes the complement of the interior of $N(\omega)$. It is also obvious that

$$\varphi(\omega, N^-(\omega)) \cap \text{int} N(\theta\omega) = \emptyset, \quad \varphi(\omega, N(\omega) \setminus N^-(\omega)) \subset \text{int} N(\theta\omega).$$

These simple facts will be used in the sequel.

The following remark is useful in our proof, see e.g. Lemmas 5.1 and 5.2.

Remark 3.2. (i) Assume $N(\omega)$ is a random compact set, it is obvious that $\text{Inv} N(\omega) \subset \Omega_N(\omega)$ and it is easy to see that $\text{Inv} N(\omega) = \Omega_N(\omega)$ if and only if $\Omega_N(\omega) \subset N(\omega)$. In particular, when $N(\omega)$ is forward invariant, we have $\Omega_N(\omega) \subset N(\omega)$.

(ii) Unlike omega-limit sets, in general we only have

$$\text{Inv} D_1(\omega) \cup \text{Inv} D_2(\omega) \subset \text{Inv}(D_1 \cup D_2)(\omega),$$

but not $\text{Inv} D_1(\omega) \cup \text{Inv} D_2(\omega) = \text{Inv}(D_1 \cup D_2)(\omega)$.

4 Random filtration pair

Definition 4.1. Assume $N(\omega)$ is a random isolating neighborhood, $L(\omega) \subset N(\omega)$ is a random compact set and $S(\omega)$ is the random isolated invariant set inside $N(\omega)$. We also assume $N(\omega) = \text{cl}(\text{int} N(\omega))$, $L(\omega) = \text{cl}(\text{int} L(\omega))$. We call $(N(\omega), L(\omega))$ is a *random filtration pair* for $S(\omega)$ if the following holds:

- $\text{cl}(N(\omega) \setminus L(\omega))$ is a random isolating neighborhood of $S(\omega)$;
- $L(\omega)$ is a random neighborhood of $N^-(\omega)$ in $N(\omega)$;
- $\varphi(\omega, L(\omega)) \cap \text{cl}(N(\theta\omega) \setminus L(\theta\omega)) = \emptyset$.

Similar to [11], for a given random variable $\epsilon(\omega) > 0$ we define a *random $\epsilon(\omega)$ -chain of length n* to be $n + 1$ random variables $x_0(\omega), x_1(\omega), \dots, x_n(\omega)$ satisfying that

$$d_X(\varphi(\theta_{-1}\omega, x_i(\theta_{-1}\omega)), x_{i+1}(\omega)) < \epsilon(\omega),$$

for $i = 0, 1, \dots, n - 1$.

Definition 4.2. Assume $N(\omega)$ is a random isolating neighborhood and $S(\omega)$ is the random isolated invariant set inside $N(\omega)$. For any random variable $\epsilon(\omega) > 0$, we define the *random $\epsilon(\omega)$ -chain neighborhood of $S(\omega)$* relative to $N(\omega)$, $C_\epsilon(N, S)(\omega)$, to be the union of the random variables $x(\omega)$ such that $\{x_i(\omega)\}_{i=0}^m \subset N(\omega)$ is a random $\epsilon(\omega)$ -chain satisfying that $x_{-l}(\omega), x_m(\omega) \in S(\omega)$ and $x_0(\omega) = x(\omega)$.

For the properties of the random $\epsilon(\omega)$ -chain neighborhood of $S(\omega)$ relative to $N(\omega)$, see the following several lemmas.

Lemma 4.1. *Assume $N(\omega)$ is a random isolating neighborhood and $S(\omega)$ is the random isolated invariant set inside $N(\omega)$. Then $C_\epsilon(N, S)(\omega)$ is a random open set relative to $N(\omega)$.*

Proof. Define

$$\begin{aligned} U_0(\omega) &:= S(\omega), \\ U_1(\omega) &:= B_{\epsilon(\omega)}(\varphi(\theta_{-1}\omega, S(\theta_{-1}\omega))) (= B_{\epsilon(\omega)}(S(\omega))) \\ U_2(\omega) &:= B_{\epsilon(\omega)}(\varphi(\theta_{-1}\omega, U_1(\theta_{-1}\omega))) \\ &\dots\dots \\ U_n(\omega) &:= B_{\epsilon(\omega)}(\varphi(\theta_{-1}\omega, U_{n-1}(\theta_{-1}\omega))) \\ &\dots\dots \end{aligned}$$

and define

$$\begin{aligned} V_0(\omega) &:= S(\omega), \\ V_1(\omega) &:= \varphi^{-1}(\theta\omega, B_{\epsilon(\theta\omega)}(V_0(\theta\omega))) \\ &\dots\dots \\ V_n(\omega) &:= \varphi^{-1}(\theta\omega, B_{\epsilon(\theta\omega)}(V_{n-1}(\theta\omega))) \\ &\dots\dots, \end{aligned}$$

where $B_r(A)$ stands for the open r -neighborhood of the set A . Denote

$$U(\omega) = \bigcup_{i=0}^{\infty} U_i(\omega), \quad V(\omega) = \bigcup_{i=0}^{\infty} V_i(\omega),$$

then by the definition of $C_\epsilon(N, S)(\omega)$ we have

$$C_\epsilon(N, S)(\omega) = U(\omega) \cap V(\omega) \cap N(\omega).$$

By the fact $\varphi(\omega, \cdot)$ is homeomorphism for $\forall \omega$, we obtain that each $U_i(\omega), V_i(\omega)$, $i = 1, 2, \dots$ is a random open set. Then we obtain that $U(\omega), V(\omega)$ are random open sets. It is easy to see that $U^c(\omega) \cup V^c(\omega)$ is measurable, i.e. the complement of $U(\omega) \cap V(\omega)$ is a random closed set. So $U(\omega) \cap V(\omega)$ is a random open set. Hence $C_\epsilon(N, S)(\omega)$ is a random open set relative to $N(\omega)$. \square

Lemma 4.2. *Assume $N(\omega)$ is a random isolating neighborhood and $S(\omega)$ is the random isolated invariant set inside $N(\omega)$. Then $S(\omega)$ can be characterized by*

$$S(\omega) = \bigcap \{\text{cl}(C_\epsilon(N, S)(\omega)) \mid \epsilon(\omega) > 0\}. \quad (3)$$

Proof. The idea of the proof is originated from Easton [5]. It is obvious that the left hand of (3) is a subset of right hand, so we only need to verify that the

converse inclusion is true almost surely. For arbitrary random variable $x(\omega) \in \bigcap \{\text{cl}(C_\epsilon(N, S)(\omega)) \mid \epsilon(\omega) > 0\}$, and $\forall \epsilon(\omega) > 0$, there exists a random $\epsilon(\omega)$ -chain in $N(\omega)$ from $x(\omega)$ to $S(\omega)$ by the definition of $C_\epsilon(N, S)(\omega)$. Let $\epsilon(\omega) \rightarrow 0$, by the invariance of $S(\omega)$ and the measure preserving of θ_n , we obtain that the entire forward orbit of $x(\omega)$ is in $N(\omega)$ almost surely, i.e. $\varphi^n(\omega, x(\omega)) \in N(\theta_n \omega)$ for $\forall n \in \mathbb{N}$. Similarly the entire backward orbit of $x(\omega)$ is also in $N(\omega)$ almost surely. Hence we have obtained

$$\varphi^n(\omega, x(\omega)) \in N(\theta_n \omega), \forall n \in \mathbb{Z},$$

i.e. $x(\omega) \in S(\omega)$ almost surely by the definition of $S(\omega)$. Thus we have verified that the right hand of (3) is a subset of $S(\omega)$ almost surely. \square

Lemma 4.3. *Assume $S(\omega)$ is a random isolated invariant set with a random isolating neighborhood $N(\omega)$ and $W(\omega)$ is an arbitrary random neighborhood of $S(\omega)$ in $N(\omega)$. Then there exists a random variable $\epsilon(\omega) > 0$ such that $C_\epsilon(N, S)(\omega) \subset W(\omega)$.*

Proof. By the invariance of $S(\omega)$, it is easy to see that for a given $k \in \mathbb{N}$, there exists a sufficiently small random variable $\tilde{\epsilon}(\omega)$ such that all the random $\tilde{\epsilon}(\omega)$ -chains with length not greater than k from $S(\omega)$ to $S(\omega)$ must be in $W(\omega)$. Denote $\epsilon_n(\omega) = \frac{1}{2^n} \tilde{\epsilon}(\omega)$, then by a similar argument to that of Lemma 4.2 we obtain that

$$S(\omega) = \bigcap_{n \in \mathbb{N}} \{\text{cl}(C_{\epsilon_n}(N, S)(\omega))\}.$$

For simplicity, we denote $S_n(\omega) = \text{cl}(C_{\epsilon_n}(N, S)(\omega))$. It is obvious that $S_n(\omega) \supset S_{n+1}(\omega)$, hence we have

$$S(\omega) = \bigcap_{n \in \mathbb{N}} S_n(\omega) = \lim_{n \rightarrow \infty} S_n(\omega).$$

Therefore, $\exists \tilde{n} = \tilde{n}(\omega)$ such that $S_n(\omega) \subset W(\omega)$ whenever $n \geq \tilde{n}$. In fact, if we take

$$\tilde{n}(\omega) := \inf \{n \in \mathbb{N} \mid d(S_n(\omega) \setminus W(\omega)) = 0\},$$

then, similar to Lemma 3.5 of [11] and by Proposition 2.3, we obtain that $\tilde{n}(\omega)$ is measurable. Define $\epsilon(\omega) := \epsilon_{\tilde{n}(\omega)}$, then $\epsilon(\omega)$ is measurable and by its definition we have $C_\epsilon(N, S)(\omega) \subset W(\omega)$. This completes the proof of the lemma. \square

Lemma 4.4. *Assume $S(\omega)$ is a random isolated invariant set with a random isolating neighborhood $N(\omega)$ and $\epsilon(\omega) > 0$ is sufficiently small, then $\text{cl}(C_\epsilon(N, S)(\omega))$ is a random isolating block.*

Proof. For $\forall x(\omega) \in C_\epsilon(N, S)(\omega)$, by the definition of $C_\epsilon(N, S)(\omega)$, we have the following holds:

$$B_{\epsilon(\omega)}(\varphi(\theta_{-1}\omega, x(\theta_{-1}\omega))) \cap N(\omega) \cap \varphi^{-1}(\theta\omega, B_{\epsilon(\theta\omega)}(x(\theta\omega))) \subset C_\epsilon(N, S)(\omega).$$

By the fact that $\varphi(\omega, \cdot)$ is a homeomorphism on X , there exists a $\delta(\omega) > 0$ such that

$$B_{\delta(\omega)}(\varphi(\theta_{-1}\omega, x(\theta_{-1}\omega))) \cap N(\omega) \cap B_{\delta(\omega)}(\varphi^{-1}(\theta\omega, x(\theta\omega))) \subset C_\epsilon(N, S)(\omega).$$

Since $x(\omega) \in C_\epsilon(N, S)(\omega)$ is arbitrary, we obtain that

$$\begin{aligned} & \varphi(\theta_{-1}\omega, \text{cl}(C_\epsilon(N, S)(\theta_{-1}\omega))) \cap N(\omega) \cap \varphi^{-1}(\theta\omega, \text{cl}(C_\epsilon(N, S)(\theta\omega))) \\ & \subset C_\epsilon(N, S)(\omega) \subset \text{int}(\text{cl}(C_\epsilon(N, S)(\omega))), \end{aligned}$$

i.e. $\text{cl}(C_\epsilon(N, S)(\omega))$ is a random isolating block. This completes the proof of the lemma. \square

Assume $\varphi(\omega, \cdot), \psi(\omega, \cdot)$ are two random homeomorphisms and $N(\omega)$ is a random compact set. We define

$$\begin{aligned} d_{\varphi, \psi}^N(\omega) &:= \sup_{x \in N(\theta_{-1}\omega)} d_X(\varphi(\theta_{-1}\omega, x), \psi(\theta_{-1}\omega, x)) \\ &+ \sup_{y \in N(\theta\omega)} d_X(\varphi^{-1}(\theta\omega, y), \psi^{-1}(\theta\omega, y)). \end{aligned}$$

Then by the measurable selection theorem we have

$$\begin{aligned} d_{\varphi, \psi}^N(\omega) &= \sup_{i=1}^{\infty} d_X(\varphi(\theta_{-1}\omega, x_i(\theta_{-1}\omega)), \psi(\theta_{-1}\omega, x_i(\theta_{-1}\omega))) \\ &+ \sup_{j=1}^{\infty} d_X(\varphi^{-1}(\theta\omega, x_j(\theta\omega)), \psi^{-1}(\theta\omega, x_j(\theta\omega))), \end{aligned} \quad (4)$$

where $\{x_n(\omega)\}_{n=1}^{\infty}$ is a family of countable dense random variables of $N(\omega)$. Hence $d_{\varphi, \psi}^N(\omega)$ is measurable. It is easy to see that for any given $\omega \in \Omega$, $d_{\varphi, \psi}^N(\omega)$ is a metric defined on the space of random homeomorphisms whose definition domain contains $N(\omega)$ and we call it “random metric”. In the following, we say a family of random homeomorphisms ψ_n approximate ψ in the random C^0 topology if $d_{\psi_n, \psi}^N(\omega)$ converge to 0 when $n \rightarrow \infty$.

The following theorem states that the random filtration pairs for a random isolated invariant set are robust under small random C^0 perturbations of the discrete random dynamical system φ , through which we can obtain the continuation property of random Conley index easily.

Theorem 4.1. *Assume $N(\omega)$ is a random isolating block and $L(\omega)$ is a sufficiently small random compact neighborhood of $N^-(\omega)$ in $N(\omega)$, then $(N(\omega), L(\omega))$ is a random filtration pair for $\text{Inv}N(\omega)$. Moreover, there exists a random C^0 -neighborhood of φ such that for any random homeomorphism ψ in this neighborhood $S_\psi(\omega) := \text{Inv}(N(\omega) \setminus L(\omega), \psi)$ is a random isolated invariant set for ψ and $(N(\omega), L(\omega))$ is a random filtration pair for $S_\psi(\omega)$.*

Proof. (1) By the definition of $N^-(\omega)$, we have

$$\varphi(\omega, N^-(\omega)) \subset \text{int}^c N(\theta\omega). \quad (5)$$

Denote $S(\omega) = \text{Inv}N(\omega)$, then we have

$$\varphi(\omega, S(\omega)) = S(\theta\omega) \subset \text{int}N(\theta\omega) \quad (6)$$

by the fact that $N(\omega)$ is a random isolating block. Since $\varphi(\omega, \cdot)$ is homeomorphism, by (5) and (6), we know that $N^-(\omega)$ and $S(\omega)$ are disjoint random compact sets. Hence when $L(\omega)$ is a sufficiently small random compact neighborhood of $N^-(\omega)$, we have $N(\omega) \setminus L(\omega)$ is a random neighborhood of $S(\omega)$. By the definitions of random isolating neighborhood and random isolated invariant set, we know that any random neighborhood $V(\omega)$ of $S(\omega)$ in $N(\omega)$ is a random isolating neighborhood with $S(\omega) = \text{Inv}V(\omega)$. In particular, $\text{cl}(N(\omega) \setminus L(\omega))$ is a random isolating neighborhood of $S(\omega)$.

Now we verify that

$$\varphi(\omega, L(\omega)) \cap \text{cl}(N(\theta\omega) \setminus L(\theta\omega)) = \emptyset.$$

By Remark 3.1 we know that

$$\varphi(\omega, N(\omega) \setminus N^-(\omega)) \subset \text{int}N(\theta\omega),$$

i.e.

$$N(\omega) \setminus N^-(\omega) \subset \varphi^{-1}(\theta\omega, \text{int}N(\theta\omega)) \cap N(\omega) \subset \varphi^{-1}(\theta\omega, N(\theta\omega)) \cap N(\omega).$$

By the closeness of $\varphi^{-1}(\theta\omega, N(\theta\omega)) \cap N(\omega)$, we have

$$\text{cl}(N(\omega) \setminus N^-(\omega)) \subset \varphi^{-1}(\theta\omega, N(\theta\omega)) \cap N(\omega). \quad (7)$$

On the other hand, by (2) we have

$$\begin{aligned} \varphi(\theta_{-1}\omega, N^-(\theta_{-1}\omega)) &= \varphi[\theta_{-1}\omega, N(\theta_{-1}\omega) \cap (\varphi^{-1}(\omega, \text{int}^c N(\omega)))] \\ &\subset [\varphi(\theta_{-1}\omega, N(\theta_{-1}\omega))] \cap [\varphi(\theta_{-1}\omega, \varphi^{-1}(\omega, \text{int}^c N(\omega)))] \\ &= \varphi(\theta_{-1}\omega, N(\theta_{-1}\omega)) \cap \text{int}^c N(\omega) \end{aligned} \quad (8)$$

By (7) and (8) we have

$$\begin{aligned} &\varphi(\theta_{-1}\omega, N^-(\theta_{-1}\omega)) \cap \text{cl}(N(\omega) \setminus N^-(\omega)) \\ &\subset \varphi(\theta_{-1}\omega, N(\theta_{-1}\omega)) \cap \text{int}^c N(\omega) \cap \varphi^{-1}(\theta\omega, N(\theta\omega)) \cap N(\omega) \\ &= \emptyset, \end{aligned}$$

where ‘=’ holds by the fact that $N(\omega)$ is a random isolating block. By the fact that $\varphi(\omega, \cdot)$ is a homeomorphism we have $\varphi(\theta_{-1}\omega, N^-(\theta_{-1}\omega))$ is a random compact set. Then when $L(\omega)$ is a sufficiently small random compact neighborhood of $N^-(\omega)$ in $N(\omega)$ we have

$$\begin{aligned} &\varphi(\theta_{-1}\omega, L(\theta_{-1}\omega)) \cap \text{cl}(N(\omega) \setminus L(\omega)) \\ &\subset \varphi(\theta_{-1}\omega, L(\theta_{-1}\omega)) \cap \text{cl}(N(\omega) \setminus N^-(\omega)) \end{aligned}$$

$$= \emptyset.$$

Up to now we have proved that $(N(\omega), L(\omega))$ is a random filtration pair for $\text{Inv}N(\omega)$.

(2) Assume ψ is a random homeomorphism sufficiently close to φ in the random C^0 topology. Then it is easy to see that

$$\psi(\theta_{-1}\omega, N(\theta_{-1}\omega)) \cap N(\omega) \cap \psi^{-1}(\theta\omega, N(\theta\omega)) \subset \text{int}N(\omega)$$

by the fact that $N(\omega)$ is a random isolating neighborhood for φ . Hence $N(\omega)$ is also a random isolating block for ψ . We denote $N_\psi^-(\omega)$ the exit set of $N(\omega)$ for ψ . In fact, it is easy to verify that $L(\omega)$ being a random neighborhood of $N^-(\omega)$ in $N(\omega)$ is equivalent to that

$$\varphi(\omega, \text{cl}(N(\omega) \setminus L(\omega))) \subset \text{int}N(\theta\omega) \quad (9)$$

by the fact $\varphi(\omega, \cdot)$ is homeomorphism again. Then we have (9) holds if we replace φ with ψ when ψ is sufficiently close to φ , which indicates that $L(\omega)$ is a random neighborhood of $N_\psi^-(\omega)$ in $N(\omega)$. Therefore $(N(\omega), L(\omega))$ is a random filtration pair for the random homeomorphisms sufficiently close to φ in the random C^0 topology. \square

Assume $P = (N(\omega), L(\omega))$ is a random filtration pair for φ and denote $N_L(\omega)$ the quotient space $N(\omega)/L(\omega)$, where the collapsed set $L(\omega)$ is denoted by $[L(\omega)]$ and is taken as the base-point. That is,

$$N_L(\omega) = (N(\omega) \setminus L(\omega) \cup [L(\omega)], [L(\omega)]).$$

For $\forall \omega \in \Omega$, a set $U(\omega) \subset N_L(\omega)$ is open if either $U(\omega)$ is open in $N(\omega)$ and $U(\omega) \cap L(\omega) = \emptyset$ or the set $(U(\omega) \cap (N(\omega) \setminus L(\omega))) \cup L(\omega)$ is open in $N(\omega)$. In the case $L(\omega) = \emptyset$, $N_L(\omega)$ is obtained from $N(\omega)$ by adding an isolated base-point, i.e. $N_L(\omega) = N(\omega) \cup [\emptyset]$. We call the random quotient space obtained in this way *random pointed space*. In particular, \emptyset/\emptyset is a random pointed space consisting of just one point—base point. In the following we will identify $N_L(\omega) \setminus [L(\omega)]$ with $N(\omega) \setminus L(\omega)$.

We define the *random pointed space map associated to P* , i.e. $\varphi_P(\omega, \cdot) : N_L(\omega) \rightarrow N_L(\theta\omega)$, as follows:

$$\varphi_P(\omega, \cdot) = \begin{cases} [L(\theta\omega)], & x = [L(\omega)] \text{ or } \varphi(\omega, x) \notin N(\theta\omega), \\ p(\theta\omega, \varphi(\omega, x)), & \text{otherwise,} \end{cases}$$

where $p(\omega, \cdot) : N(\omega) \rightarrow N_L(\omega)$ is the quotient map. Then we have the following theorem.

Theorem 4.2. *The random pointed space map associated to P , $\varphi_P(\omega, \cdot) : N_L(\omega) \rightarrow N_L(\theta\omega)$ is a base-point preserving random map with the property that $[L(\omega)] \subset \text{int}\varphi_P^{-1}(\theta\omega, [L(\theta\omega)])$. Moreover, $\varphi_P(\omega, \cdot) : N_L(\omega) \rightarrow N_L(\theta\omega)$ is continuous, and $\varphi_P(\cdot, x)$ is measurable.*

Proof. By the definition of φ_P it is obvious that φ_P preserves the base-point. Since $P = (N(\omega), L(\omega))$ is a random filtration pair, we have

$$\varphi(\omega, L(\omega)) \cap \text{cl}(N(\theta\omega) \setminus L(\theta\omega)) = \emptyset.$$

Hence by the compactness of $L(\omega)$ and $\text{cl}(N(\theta\omega) \setminus L(\theta\omega))$, there exists a random neighborhood $K(\omega)$ of $L(\omega)$ such that

$$\varphi(\omega, K(\omega)) \cap \text{cl}(N(\theta\omega) \setminus L(\theta\omega)) = \emptyset.$$

Therefore by the definition of φ_P we have $\varphi_P(\omega, x) = [L(\theta\omega)]$ whenever $x \in K(\omega)$, i.e. $[L(\omega)] \subset \text{int}\varphi_P^{-1}(\theta\omega, [L(\theta\omega)])$.

It is obvious that when x satisfies $\varphi(\omega, x) \notin N(\theta\omega)$ there exists a small neighborhood $N_x(\omega)$ of x such that $\varphi(\omega, N_x(\omega)) \cap N(\theta\omega) = \emptyset$. That is, $\varphi_P(\omega, \cdot)$ is continuous at x . When $\varphi_P(\omega, x) = p(\theta\omega, \varphi(\omega, x))$, $\varphi_P(\omega, \cdot)$ is the composition of two continuous functions $p(\theta\omega, \cdot)$ and $\varphi(\omega, \cdot)$, so it is continuous. Hence we only need to verify that $\varphi(\omega, \cdot)$ is continuous at $[L(\omega)]$. Assume x_n is a sequence converging to $[L(\omega)]$, then when n is sufficiently large we have $x \in K(\omega)$. Hence we have

$$\varphi(\omega, x_n) = [L(\theta\omega)] = \varphi(\omega, [L(\omega)])$$

when n is sufficiently large, which verifies that $\varphi(\omega, \cdot)$ is continuous at $[L(\omega)]$.

Since $\varphi(\cdot, x)$ is measurable, we immediately obtain that $\varphi_P(\cdot, x)$ is measurable by its definition. \square

Remark 4.1. Similar to deterministic case, to study the dynamical behaviors of dynamical systems, the definitions of forward invariant, backward invariant and invariant sets are introduced for RDS, see [1] for details. Based on this, for a given random set we can define the exit set of it, the maximal invariant random set in it etc. These definitions are very natural and have their obvious dynamical explanations. If we go one step further, inspired by [6], we can introduce the definition of random filtration pair, which is crucial to define Conley index for RDS. Of course the random filtration pair for an isolated invariant set is important to study the dynamical behaviors for RDS.

5 Random shift equivalence

In order to define the random Conley index for discrete random dynamical systems, we need to find an invariant for random isolated invariant sets–random shift equivalence.

Assume $C(\omega), D(\omega)$ are two random pointed spaces and

$$c(\omega, \cdot) : C(\omega) \rightarrow C(\theta\omega), \quad d(\omega, \cdot) : D(\omega) \rightarrow D(\theta\omega)$$

are two base-point preserving maps satisfying that $c(\omega, \cdot), d(\omega, \cdot)$ are continuous and $c(\cdot, x), d(\cdot, x)$ are measurable. If there exist $r(\omega, \cdot) : C(\omega) \rightarrow D(\theta_{n_1}\omega)$, $s(\omega, \cdot) : D(\omega) \rightarrow C(\theta_{n_2}\omega)$, where r, s have the same properties as c, d (i.e. they

satisfy the continuous-measurable condition and they preserve base-point) and $n_i = n_i(\omega), i = 1, 2$ are measurable, such that the following diagrams

$$\begin{array}{ccccc}
C(\omega) & \xrightarrow{c(\omega, \cdot)} & C(\theta\omega) & & D(\omega) & \xrightarrow{d(\omega, \cdot)} & D(\theta\omega) \\
r(\omega, \cdot) \downarrow & & \text{(I)} \downarrow r(\theta\omega, \cdot) & & s(\omega, \cdot) \downarrow & & \text{(III)} \downarrow s(\theta\omega, \cdot) \\
D(\theta_{n_1}\omega) & \xrightarrow[\text{(II)}]{d(\theta_{n_1}\omega, \cdot)} & D(\theta_*\omega) & & C(\theta_{n_2}\omega) & \xrightarrow[\text{(IV)}]{c(\theta_{n_2}\omega, \cdot)} & C(\theta_*\omega)
\end{array} \quad (10)$$

are *quasi-commutative* and the following holds:

$$r(\theta_{n_2}\omega, s(\omega, \cdot)) = d^{n_2(\omega) + n_1(\theta_{n_2}\omega)}(\omega, \cdot), \quad (11)$$

$$s(\theta_{n_1}\omega, r(\omega, \cdot)) = c^{n_1(\omega) + n_2(\theta_{n_1}\omega)}(\omega, \cdot). \quad (12)$$

Then we call (C, c) is *random shift equivalent* to (D, d) and denote it by $(C, c) \sim (D, d)$. We use “quasi-commutative” meaning that the two diagrams in (10) are not strictly commutative. For example, in the first diagram of (10) we obviously have

$$\begin{aligned}
d(\theta_{n_1}\omega, r(\omega, \cdot)) &: C(\omega) \rightarrow D(\theta_{n_1(\omega)+1}\omega), \\
r(\theta\omega, c(\omega, \cdot)) &: C(\omega) \rightarrow D(\theta_{n_1(\theta\omega)+1}\omega).
\end{aligned}$$

That is, (I) and (II) do not generally arrive at the same destination unless we have $n_1(\omega) = n_1(\theta\omega)$. But in the situation we confront, this condition is not satisfied generally. Hence the diagram does not commute generally. But if we can adjust it to make the diagram commute in the following sense, i.e. if we have

$$\begin{cases} r(\theta\omega, c(\omega, \cdot)) = d^{n_1(\theta\omega) - n_1(\omega)}(\theta_{n_1(\omega)+1}\omega, d(\theta_{n_1(\omega)}\omega, r(\omega, \cdot))), & n_1(\theta\omega) \geq n_1(\omega), \\ d^{n_1(\omega) - n_1(\theta\omega)}(\theta_{n_1(\theta\omega)+1}\omega, r(\theta\omega, c(\omega, \cdot))) = d(\theta_{n_1(\omega)}\omega, r(\omega, \cdot)), & n_1(\theta\omega) < n_1(\omega), \end{cases} \quad (13)$$

then we call the diagram quasi-commutative. For simplicity, we can formally rewrite (13) as

$$r(\theta\omega, c(\omega, \cdot)) = d^\Delta(\theta_{n_1(\omega)+1}\omega, d(\theta_{n_1(\omega)}\omega, r(\omega, \cdot))),$$

where d^Δ denotes the adjustment we make and it should be understood as in (13), i.e. d^Δ should vary according to the relation $n_1(\theta\omega) \geq n_1(\omega)$ or $n_1(\theta\omega) < n_1(\omega)$. Symmetrically we also can rewrite (13) as

$$d^\Delta(\theta_{n_1(\theta\omega)+1}\omega, r(\theta\omega, c(\omega, \cdot))) = d(\theta_{n_1(\omega)}\omega, r(\omega, \cdot)).$$

And in the first diagram of (10), $D(\theta_*\omega)$ means that through (I), (II) we may arrive at different destinations, i.e. $D(\theta_{n_1(\theta\omega)+1}\omega)$ and $D(\theta_{n_1(\omega)+1}\omega)$. By the above adjustment we write $D(\theta_*\omega)$ to indicate that the diagram is quasi-commutative and $\theta_*\omega$ should be valued appropriately, i.e. $* = \max\{n_1(\omega) + 1, n_1(\theta\omega) + 1\}$. Loosely speaking, starting from the same start point $(C(\omega))$, we

can (by adjustment) arrive at the same end point ($D(\theta_*\omega)$) through two paths ($r \circ c$ and $d \circ r$). This is just our intuitive idea of quasi-commutation. In the following for simplicity we often use “*” to denote the number we can obviously understand but we have no need to formulate it explicitly. The second diagram in (10) can be defined quasi-commutative in the completely same way. In (11), we only care that the composition of r and s equals the power of d , but we do not care what on earth the power is in spite that we have explicitly given it. Hence for simplicity, we often formally rewrite (11) as $r \circ s = d^*$ and rewrite (12) as $s \circ r = c^*$.

We now verify that the random shift equivalence is an equivalence relation. The reflexivity and symmetry obviously hold and we only need to verify the transitivity of random shift equivalence. Assume (C, c) is random shift equivalent to (D, d) with (10), (11) and (12) hold and assume (D, d) is random shift equivalent to (E, e) with

$$\begin{aligned} r_1(\theta\omega, d(\omega, \cdot)) &= e^{\Delta}(\theta_{n'_1(\omega)+1}\omega, e(\theta_{n'_1(\omega)}\omega, r_1(\omega, \cdot))), \\ s_1(\theta\omega, e(\omega, \cdot)) &= d^{\Delta}(\theta_{n'_2(\omega)+1}\omega, d(\theta_{n'_2(\omega)}\omega, s_1(\omega, \cdot))), \\ r_1(\theta_{n'_2}\omega, s_1(\omega, \cdot)) &= e^{n'_2(\omega)+n'_1(\theta_{n'_2}\omega)}(\omega, \cdot), \\ s_1(\theta_{n'_1}\omega, r_1(\omega, \cdot)) &= d^{n'_1(\omega)+n'_2(\theta_{n'_1}\omega)}(\omega, \cdot), \end{aligned}$$

where

$$r_1(\omega, \cdot) : D(\omega) \rightarrow E(\theta_{n'_1}\omega), \quad s_1(\omega, \cdot) : E(\omega) \rightarrow D(\theta_{n'_2}\omega).$$

If we denote $r_2 = r_1 \circ d^{\Delta} \circ r$, then we have

$$\begin{aligned} C(\omega) &\xrightarrow{d^{\Delta} \circ r \circ c} D(\theta_*\omega) \xrightarrow{r_1} E(\theta_{[n'_1(\theta_*\omega)+*]}\omega), \\ C(\omega) &\xrightarrow{d^{\Delta} \circ r} D(\theta_*\omega) \xrightarrow{e \circ r_1} E(\theta_{[n'_1(\theta_*\omega)+*+1]}\omega), \end{aligned}$$

that is

$$\begin{aligned} r_2 \circ c : C(\omega) &\rightarrow E(\theta_{[n'_1(\theta_*\omega)+*]}\omega), \\ e \circ r_2 : C(\omega) &\rightarrow E(\theta_{[n'_1(\theta_*\omega)+*+1]}\omega). \end{aligned}$$

By the assumption $(C, c) \sim (D, d)$ and $(D, d) \sim (E, e)$ it is easy to see that the following diagram

$$\begin{array}{ccc} C(\omega) & \xrightarrow{c(\omega, \cdot)} & C(\theta\omega) \\ r_2(\omega, \cdot) \downarrow & & \downarrow r_2(\theta\omega, \cdot) \\ E(\theta_{n''_1}\omega) & \xrightarrow{e(\theta_{n''_1}\omega, \cdot)} & E(\theta_*\omega) \end{array}$$

is quasi-commutative, where $n''_1 = n'_1(\theta_*\omega) + *$ and $E(\theta_*\omega)$ should be understood similar to $D(\theta_*\omega)$ above. Let $s_2 = s \circ d^{\Delta} \circ s_1$, in the completely same way we can obtain that

$$s_2(\theta\omega, e(\omega, \cdot)) = c^{\Delta}(\theta_{n''_2(\omega)+1}\omega, c(\theta_{n''_2(\omega)}\omega, s_2(\omega, \cdot))),$$

that is, the diagram

$$\begin{array}{ccc} E(\omega) & \xrightarrow{e(\omega, \cdot)} & E(\theta\omega) \\ s_2(\omega, \cdot) \downarrow & & \downarrow s_2(\theta\omega, \cdot) \\ C(\theta_{n_2}''\omega) & \xrightarrow{c(\theta_{n_2}''\omega, \cdot)} & C(\theta_*\omega) \end{array}$$

is quasi-commutative. Now we verify that $r_2 \circ s_2 = e^*$. Notice that

$$\begin{aligned} r_2 \circ s_2(\omega, \cdot) &= r_1 \circ d^\Delta \circ (r \circ s) \circ d^\Delta \circ s_1(\omega, \cdot) = r_1 \circ d^* \circ s_1(\omega, \cdot) \\ &= (r_1 \circ d) \circ d^{*-1} \circ s_1(\omega, \cdot) = (e^\Delta \circ e \circ r_1) \circ d^{*-1} \circ s_1(\omega, \cdot) \\ &= e^* \circ (r_1 \circ d) \circ d^{*-2} \circ s_1(\omega, \cdot) = \dots \\ &= e^* \circ r_1 \circ s_1(\omega, \cdot) = e^*(\omega, \cdot), \end{aligned}$$

where “*” is not an invariant number. It is only a number we obviously understand, and in our eyes we do not distinguish $*$, $* + \Delta$, $\Delta + 1$, $* + *$ etc and hence we denote them by unified notation “*”. In the same way we can obtain $s_2 \circ r_2 = c^*$. Hence we have verified $(C, c) \sim (E, e)$, i.e. random shift equivalence is an equivalence relation.

Lemma 5.1. *Assume $P' = (N(\omega), L'(\omega))$ and $P = (N(\omega) \cup L(\omega), L(\omega))$ are two random filtration pairs for $S(\omega)$ and that $L'(\omega) \subset L(\omega)$ and $\varphi(\omega, L(\omega)) \subset \text{int}L(\theta\omega)$. Then the induced random maps, $\varphi_{P'}$ and φ_P are random shift equivalent.*

Proof. Denote $Q(\omega) = N(\omega) \cup L(\omega)$ and define $r(\omega, \cdot) : N_{L'}(\omega) \rightarrow Q_L(\omega)$ as follows:

$$r(\omega, \cdot) = \begin{cases} [L(\omega)], & x = [L'(\omega)], \\ p(\omega, x), & \text{otherwise,} \end{cases}$$

where $p(\omega, \cdot) : Q(\omega) \rightarrow Q_L(\omega)$ is the quotient map. It is easy to see that $r(\omega, \cdot)$ is continuous, $r(\cdot, x)$ is measurable and $r(\theta\omega, \varphi_{P'}(\omega, \cdot)) = \varphi_P(\omega, r(\omega, \cdot))$.

We affirm that for almost all $\omega \in \Omega$, $\exists n = n(\omega)$ such that $\varphi^k(\omega, N(\omega) \cap L(\omega)) \subset L'(\theta_k\omega)$ for some $k < n$. (In fact, similar to Lemma 4.3, $n(\omega)$ can be chosen measurable.) If the assertion is false, we have

$$\mathbb{P}\{\omega \mid \exists x \in N(\omega) \cap L(\omega) \text{ such that } \varphi^k(\omega, x) \in (N(\theta_k\omega) \setminus L'(\theta_k\omega)), \forall k \in \mathbb{N}\} > 0$$

by the fact $L'(\omega)$ is a random neighborhood of $N^-(\omega)$ in $N(\omega)$. Denote this set $\tilde{\Omega}$, so $\mathbb{P}(\tilde{\Omega}) > 0$. For $\forall \omega \in \tilde{\Omega}$, let

$$\tilde{L}(\omega) := \left(\bigcap_{k \geq 0} \varphi^{-k}(\theta_k\omega, N(\theta_k\omega) \setminus L'(\theta_k\omega)) \right) \cap (N(\omega) \cap L(\omega)).$$

Then we have $\tilde{L}(\omega) \neq \emptyset$ whenever $\omega \in \tilde{\Omega}$. Let

$$\tilde{N}(\omega) := \begin{cases} \tilde{L}(\omega), & \omega \in \tilde{\Omega}, \\ S(\omega), & \omega \in \Omega \setminus \tilde{\Omega}. \end{cases}$$

By the definition of $\tilde{N}(\omega)$ we have

$$\varphi^k(\omega, \tilde{N}(\omega)) \subset N(\theta_k \omega) \setminus L'(\theta_k \omega), \forall \omega \in \Omega, \forall k \in \mathbb{N}.$$

Hence $\Omega_{\tilde{N}}(\omega)$ is an invariant random compact set in $\text{cl}(N(\omega) \setminus L'(\omega))$. By the fact that $S(\omega)$ is the maximal invariant random compact set in $\text{cl}(N(\omega) \setminus L'(\omega))$ we have $\Omega_{\tilde{N}}(\omega) \subset S(\omega)$ almost surely. But on the other hand, by the fact $\varphi(\omega, L(\omega)) \subset \text{int} L(\theta \omega)$ and measure preserving of θ_n we have

$$\begin{aligned} \mathbb{P}\{\omega \mid \Omega_{\tilde{N}}(\omega) \subset L(\omega)\} &\geq \mathbb{P}\{\omega \mid \varphi^k(\theta_{-k} \omega, \tilde{N}(\theta_{-k} \omega)) \subset L(\omega), \forall k \in \mathbb{N}\} \\ &= \mathbb{P}\{\omega \mid \varphi^k(\omega, \tilde{N}(\omega)) \subset L(\theta_k \omega), \forall k \in \mathbb{N}\} \\ &\geq \mathbb{P}\{\omega \mid \tilde{N}(\omega) \subset L(\omega)\} \\ &= \mathbb{P}(\tilde{\Omega}) > 0, \end{aligned}$$

a contradiction to the fact $L(\omega) \cap S(\omega) = \emptyset$ for $\forall \omega \in \Omega$.

For the above $n(\omega)$, if $n(\omega) \geq n(\theta \omega)$, we do not make adjustment; if $n(\omega) < n(\theta \omega)$, let $n(\omega) = n(\theta \omega)$. By such adjustment, $n(\omega)$ is still measurable and it satisfies $n(\omega) \geq n(\theta \omega)$ and $\varphi^k(\omega, N(\omega) \cap L(\omega)) \subset L'(\theta_k \omega)$ for some $k < n(\omega)$, and we still denote it by $n = n(\omega)$.

Noting that $Q(\omega) \setminus L(\omega) = N(\omega) \setminus L(\omega)$, we define $s(\omega, \cdot) : Q_L(\omega) \rightarrow N_{L'}(\theta_n \omega)$ as follows:

$$s(\omega, \cdot) = \begin{cases} [L'(\theta_n \omega)], & x = [L(\omega)], \\ \varphi_{P'}^n(\omega, p'(\omega, x)), & \text{otherwise,} \end{cases}$$

where $p'(\omega, \cdot) : N(\omega) \rightarrow N_{L'}(\omega)$ is the quotient map. It is easy to see that $s(\theta \omega, \varphi_P(\omega, \cdot)) = \varphi_{P'}^{\triangle}(\theta_{n+1} \omega, \varphi_{P'}(\theta_n \omega, s(\omega, \cdot)))$.

It is obvious that $s(\omega, \cdot)$ is continuous on $Q_L(\omega) \setminus [L(\omega)]$ since it is the composition of continuous functions. By the fact $\varphi(\omega, \cdot)$ is homeomorphism there exists a random neighborhood $V(\omega)$ of $L(\omega)$ such that $\varphi(\omega, V(\omega)) \subset \text{int} L(\theta \omega)$. Hence for $\forall x \in V(\omega)$, by the fact $n(\omega) \geq n(\theta \omega)$ we have $\varphi^k(\omega, x) \in L'(\theta_k \omega)$ for some $k \leq n$ and therefore $\varphi_{P'}^n(\omega, p'(\omega, x)) = [L'(\theta_n \omega)]$. Thus we have obtained $s(\omega, x)$ is continuous at $[L(\omega)]$. The measurability of $s(\cdot, x)$ is obvious.

By the definitions of r, s , it is easy to verify that

$$s(\omega, r(\omega, \cdot)) = \varphi_{P'}^n(\omega, \cdot), \quad r(\theta_n \omega, s(\omega, \cdot)) = \varphi_P^n(\omega, \cdot).$$

This completes the proof of the lemma. \square

Lemma 5.2. Assume $P = (N(\omega), L(\omega))$ and $P' = (N'(\omega), L(\omega))$ are two random filtration pairs for $S(\omega)$ and that $N(\omega) \subset N'(\omega)$ and $\varphi(\omega, L(\omega)) \subset \text{int} L(\theta \omega)$. Then the induced random maps, φ_P and $\varphi_{P'}$ are random shift equivalent.

Proof. Define $r(\omega, \cdot) : N_L(\omega) \rightarrow N_{L'}(\omega)$ as follows:

$$r(\omega, \cdot) = \begin{cases} [L(\omega)], & x = [L(\omega)], \\ p'(\omega, x), & \text{otherwise,} \end{cases}$$

where $p'(\omega, \cdot) : N'(\omega) \rightarrow N'_L(\omega)$ is the quotient map. It is easy to see that $r(\omega, \cdot)$ is continuous, $r(\cdot, x)$ is measurable and $r(\theta\omega, \varphi_P(\omega, \cdot)) = \varphi_{P'}(\omega, r(\omega, \cdot))$.

Since $L(\omega)$ is a random neighborhood of $N^-(\omega)$, we have $\varphi(\omega, N(\omega) \setminus L(\omega)) \subset \text{int}N(\theta\omega)$. This together with the assumption $\varphi(\omega, L(\omega)) \subset \text{int}L(\theta\omega)$ imply that $\varphi(\omega, N(\omega)) \subset \text{int}N(\theta\omega)$, i.e. $N(\omega)$ is forward invariant. Hence we have $\Omega_N(\omega) \subset \text{int}N(\omega)$ and it is the maximal invariant random compact set in $N(\omega)$.

We can decompose the random set $N'(\omega) \setminus N(\omega)$ into two random sets: $N_1(\omega)$ and $N_2(\omega)$. They are determined as follows:

$$\begin{aligned} N_1(\omega) &:= \{x \mid x \in N'(\omega) \setminus N(\omega), \varphi^k(\omega, x) \in N'(\theta_k\omega) \setminus L(\theta_k\omega), \text{ for } \forall k \in \mathbb{N}\}, \\ N_2(\omega) &:= \{x \mid x \in N'(\omega) \setminus N(\omega), \exists k \in \mathbb{N} \text{ such that } \varphi^k(\omega, x) \in L(\theta_k\omega)\}. \end{aligned}$$

By the definition of $N_1(\omega)$ and the measure preserving of θ_n we obtain that $\Omega_{N_1}(\omega) \subset \text{cl}(N'(\omega) \setminus L(\omega))$. By the definition of $N_2(\omega)$ and the forward invariance of $L(\omega)$ we have $\Omega_{N_2}(\omega) \subset L(\omega)$. Since $S(\omega), \Omega_L(\omega)$ are maximal invariant random set in $\text{cl}(N'(\omega) \setminus L(\omega)), L(\omega)$ respectively, we obtain that

$$\Omega_{N' \setminus N}(\omega) = \Omega_{N_1}(\omega) \cup \Omega_{N_2}(\omega) \subset S(\omega) \cup \Omega_L(\omega) \subset \text{int}N(\omega).$$

Therefore,

$$\Omega_{N'}(\omega) = \Omega_N(\omega) \cup \Omega_{N' \setminus N}(\omega) \subset \text{int}N(\omega).$$

Hence for $\forall \omega \in \Omega$, $\exists n(\omega)$ (in fact $n(\omega)$ may be chosen measurable similar to Lemma 4.3) such that

$$\varphi^n(\theta_{-n}\omega, N'(\theta_{-n}\omega)) \subset \text{int}N(\omega) \text{ whenever } n \geq n(\omega) - 1$$

by the forward invariance of $N'(\omega)$ (the forward invariance of $N'(\omega)$ follows completely similar to the proof of forward invariance of $N(\omega)$). In fact, it is easy to verify that for arbitrary forward invariant random set $D(\omega)$, we have

$$\varphi^n(\theta_{-n}\omega, D(\theta_{-n}\omega)) \subset \varphi^m(\theta_{-m}\omega, D(\theta_{-m}\omega)), \text{ whenever } n \geq m.$$

Then by the measure preserving of θ_n we have

$$\begin{aligned} &\mathbb{P}\{\omega \mid \varphi^n(\omega, N'(\omega)) \subset \text{int}N(\theta_n\omega), n \geq n(\omega)\} \\ &= \mathbb{P}\{\omega \mid \varphi^n(\theta_{-n}\omega, N'(\theta_{-n}\omega)) \subset \text{int}N(\omega), n \geq n(\omega)\} \\ &= 1. \end{aligned}$$

Adjust the above $n(\omega)$ similar to in Lemma 5.1 if necessary such that $n(\omega) \geq n(\theta\omega)$ and still denote it by $n = n(\omega)$. Define $s(\omega, \cdot) : N'_L(\omega) \rightarrow N_L(\theta_n\omega)$ as follows:

$$s(\omega, \cdot) = \begin{cases} [L(\theta_n\omega)], & x = [L(\omega)], \\ p(\theta_n\omega, \varphi^n(\omega, x)), & \text{otherwise,} \end{cases}$$

where $p(\omega, \cdot) : N(\omega) \rightarrow N_L(\omega)$ is the quotient map. It is easy to see that $s(\theta\omega, \varphi_{P'}(\omega, \cdot)) = \varphi_P^\Delta(\theta_{n+1}\omega, \varphi_P(\theta_n\omega, s(\omega, \cdot)))$. It is obvious that $s(\omega, \cdot)$ is continuous on $N'_L(\omega) \setminus [L(\omega)]$ since it is the composition of continuous functions.

By the assumption $\varphi(\omega, L(\omega)) \subset \text{int}L(\theta\omega)$ there exists a random neighborhood $V(\omega)$ of $L(\omega)$ such that $\varphi(\omega, V(\omega)) \subset \text{int}L(\theta\omega)$. Then by the fact $n(\omega) \geq n(\theta\omega)$ we have $s(\omega, V(\omega)) = [L(\theta_n\omega)]$, hence $s(\omega, \cdot)$ is continuous at $[L(\omega)]$. The measurability of $s(\cdot, x)$ is obvious.

By the definitions of r, s , it is easy to verify that

$$r(\theta_n\omega, s(\omega, \cdot)) = \varphi_{P'}^n(\omega, \cdot), \quad s(\omega, r(\omega, \cdot)) = \varphi_P^n(\omega, \cdot).$$

This completes the proof of the lemma. \square

By the above two lemmas we can now show that the random shift equivalent class of random pointed space maps is an invariant for a given random isolated invariant set, which is of crucial importance for the definition of random Conley index.

Theorem 5.1. *Assume $P = (N(\omega), L(\omega))$ and $P' = (N'(\omega), L'(\omega))$ are two random filtration pairs for $S(\omega)$, then the induced random maps, φ_P and $\varphi_{P'}$ on the corresponding random pointed spaces, are random shift equivalent.*

Proof. By Lemmas 4.3 and 4.4 we can choose $\epsilon(\omega)$ sufficiently small such that

$$\text{cl}(C_\epsilon(N \setminus L, S)(\omega)) = \text{cl}(C_\epsilon(N' \setminus L', S)(\omega)) \subset \text{int}(N(\omega) \setminus L(\omega)) \cap \text{int}(N'(\omega) \setminus L'(\omega)).$$

Denote $B(\omega) := \text{cl}(C_\epsilon(N \setminus L, S)(\omega))$ and assume $B_0(\omega)$ is a sufficiently small random neighborhood of $B^-(\omega)$ in $B(\omega)$, then $P_0 = (B(\omega), B_0(\omega))$ is random filtration pair for $S(\omega)$ by Theorem 4.1. We will verify that φ_{P_0} is random shift equivalent to φ_P and $\varphi_{P'}$. Since the proof is the same we only need to prove φ_{P_0} is random shift equivalent to φ_P .

We first prove that for $\forall \omega \in \Omega$, $\exists n = n(\omega)$ such that

$$\varphi_P^n(\omega, B^-(\omega)) = [L(\theta_n\omega)]. \quad (14)$$

Since $B(\omega) \subset N(\omega) \setminus L(\omega)$ and we identify $N_L(\omega) \setminus [L(\omega)]$ with $N(\omega) \setminus L(\omega)$, we may consider $B(\omega)$ as a subset of $N_L(\omega) \setminus [L(\omega)]$ and consider φ to be φ_P . By the definition of $B^-(\omega)$, for arbitrary random variable $x(\omega) \in B^-(\omega)$, there is no random $\epsilon(\omega)$ -chain from $\varphi_P(\theta_{-1}\omega, x(\theta_{-1}\omega))$ to $S(\omega)$ with positive probability. Since if there is one such $\epsilon(\omega)$ -chain, then we obtain that $\varphi_P(\theta_{-1}\omega, x(\theta_{-1}\omega)) \subset C_\epsilon(N \setminus L, S)(\omega)$ with positive probability, a contradiction to the definition of $B^-(\omega)$. Hence we have $\Omega_{B^-}(\omega) \cap S(\omega) = \emptyset$ almost surely. By the fact $\varphi_P(\omega, \cdot) : N_L(\omega) \rightarrow N_L(\theta\omega)$ and the compactness of $N_L(\omega)$ we obtain that $\Omega_{B^-}(\omega) \neq \emptyset$ almost surely and $\Omega_{B^-}(\omega) \subset N_L(\omega)$. It is easy to see that there is two invariant random set in $N_L(\omega)$ under the iteration of φ_P : $S(\omega)$ and $\{[L(\omega)]\}$. Hence we obtain that $\Omega_{B^-}(\omega) = \{[L(\omega)]\}$. By Theorem 4.2 we have $[L(\omega)] \subset \text{int}\varphi_P^{-1}(\theta\omega, [L(\theta\omega)])$. This together with the measure preserving of θ_n imply that for almost all $\omega \in \Omega$, $\exists n(\omega)$ such that

$$\varphi_P^n(\omega, B^-(\omega)) \subset \tilde{L}(\theta_n\omega), \quad n \geq n(\omega), \quad (15)$$

where $\tilde{L}(\omega) := \text{int}\varphi_P^{-1}(\theta\omega, [L(\theta\omega)])$. Hence we have verified (14). Since $\tilde{L}(\omega)$ is forward invariant under the iteration of φ_P , similar to Lemma 5.2, $n(\omega)$ in (14) can be chosen measurable.

Since $B_0(\omega)$ is a sufficiently small random neighborhood of $B^-(\omega)$, by (15) we obtain that

$$\varphi_P^n(\omega, B_0(\omega)) = [L(\theta_n\omega)], \quad (16)$$

where $n = n(\omega)$ measurable. Adjust this $n(\omega)$ similar to in Lemmas 5.1 and 5.2 such that $n(\omega) \leq n(\theta\omega)$ and still denote it by $n = n(\omega)$. Let

$$K(\omega) := \text{cl}(\text{int}\varphi_P^{-n}(\theta_n\omega, [L(\theta_n\omega)])) \cap N_L(\omega).$$

Then by the fact $[L(\omega)] \subset \text{int}\varphi_P^{-1}(\theta\omega, [L(\theta\omega)])$ and the fact $n(\omega) \leq n(\theta\omega)$, we obtain that $\varphi_P(\omega, K(\omega)) \subset \text{int}K(\theta\omega)$. Moreover, by the definition of $K(\omega)$ we have $B_0(\omega) \subset \text{int}K(\omega)$.

Let $Q = (B(\omega) \cup K(\omega), K(\omega))$, $R = (N_L(\omega), K(\omega))$, then it is easy to see that Q, R are random filtration pairs for $S(\omega)$. By Lemma 5.1 we obtain that $\varphi_{P_0} \sim \varphi_Q$ and by Lemma 5.2 we have $\varphi_Q \sim \varphi_R$. Hence $\varphi_{P_0} \sim \varphi_R$.

Let $\tilde{R} = (N(\omega), p^{-1}(\omega, K(\omega)))$, where $p(\omega, \cdot) : N(\omega) \rightarrow N_L(\omega)$ is the random quotient map. Then by Lemma 5.1 we obtain that $\varphi_P \sim \varphi_{\tilde{R}}$. Noting that we may identify the random pointed spaces and the corresponding random pointed space maps of the random filtration pairs \tilde{R} and R , i.e. $\varphi_R \sim \varphi_{\tilde{R}}$. Hence $\varphi_P \sim \varphi_R$. Therefore we have obtained $\varphi_{P_0} \sim \varphi_P$. This completes the proof of the theorem. \square

Remark 5.1. To obtain an invariant for a given isolated invariant set, parallel to [6], we introduced the definition of random shift equivalence. In fact, it can be regarded as a generalization of conjugacy of two RDSs. (For the definition of conjugacy of two RDSs, we refer to [1, 7, 8, 9, 10] for details and related applications.) To see this, notice that in the definition of random shift equivalence, in particular if we have $r(\omega, \cdot) : C(\omega) \rightarrow D(\omega)$ and $s(\omega, \cdot) : D(\omega) \rightarrow C(\omega)$, then $rs = \text{Id}$ and r (or s) play the role of cohomological random homeomorphism between c, d , i.e. c is conjugate to d through r (or s).

As pointed out in [6], the random shift equivalence constructed in Theorem 5.1 is not unique, here we only give one relative simple construction. In the definition of random shift equivalence, for simplicity, we have required that the random maps r and s preserve base-point, which of course is not necessary. In other words, the definition of random shift equivalence can be made slightly more general as that given in [6] for deterministic case.

6 Definition of Conley index for RDS

Assume $S(\omega)$ is a random isolated invariant set and $P = (N(\omega), L(\omega))$ is a random filtration pair for $S(\omega)$, by Theorem 5.1, we know that the random shift equivalent class of $\varphi_P : N_L(\omega) \rightarrow N_L(\theta\omega)$ is an invariant for $S(\omega)$. It is well-known that continuation of Conley index is one of its most important

properties, hence before defining a random Conley index we need to explain what random continuation is. Assume $C(\omega)$ is a random pointed space and $f(\omega, \cdot) : C(\omega) \rightarrow C(\theta_n \omega)$, where $n = n(\omega)$ is measurable, is a base-point preserving random map such that $f(\omega, \cdot)$ is continuous and $f(\cdot, x)$ is measurable. Let $g(\omega, \cdot) : C(\omega) \rightarrow C(\theta_n \omega)$ is another random map with the same property as that of f . We call f is *random homotopic to g* , denoted by $f \simeq g$, if there exist $H : [0, 1] \times \Omega \times C(\omega) \rightarrow C(\theta_n \omega)$ satisfying that $H(\cdot, \omega, \cdot)$ is continuous and $H(t, \cdot, x)$ is measurable. Moreover, H satisfies

$$\begin{cases} H(0, \cdot, \cdot) = f(\cdot, \cdot), \\ H(1, \cdot, \cdot) = g(\cdot, \cdot). \end{cases}$$

It is easy to see that random homotopy is an equivalence relation. We denote $[f]$ the random homotopy class with f the representative element.

Assume $C(\omega)$ and $D(\omega)$ are two random pointed spaces and

$$[c(\omega, \cdot)] : C(\omega) \rightarrow C(\theta \omega), \quad [d(\omega, \cdot)] : D(\omega) \rightarrow D(\theta \omega)$$

are random homotopy classes. If there exist random homotopy classes

$$[r(\omega, \cdot)] : C(\omega) \rightarrow D(\theta_{n_1} \omega), \quad [s(\omega, \cdot)] : D(\omega) \rightarrow C(\theta_{n_2} \omega),$$

where $n_i = n_i(\omega)$, $i = 1, 2$ are measurable, such that

$$\begin{aligned} r(\theta \omega, c(\omega, \cdot)) &\simeq d^\Delta(\theta_{n_1(\omega)+1} \omega, d(\theta_{n_1(\omega)} \omega, r(\omega, \cdot))), \\ s(\theta \omega, d(\omega, \cdot)) &\simeq c^\Delta(\theta_{n_2(\omega)+1} \omega, c(\theta_{n_2(\omega)} \omega, s(\omega, \cdot))), \\ r(\theta_{n_2} \omega, s(\omega, \cdot)) &\simeq d^{n_2(\omega)+n_1(\theta_{n_2} \omega)}(\omega, \cdot), \\ s(\theta_{n_1} \omega, r(\omega, \cdot)) &\simeq c^{n_1(\omega)+n_2(\theta_{n_1} \omega)}(\omega, \cdot), \end{aligned}$$

where d^Δ, c^Δ denote the adjustment we make, then we call the random homotopy classes $[C, c]$ and $[D, d]$ are random shift equivalent.

Now we are in the position to give the definition of random Conley index for random isolated invariant sets.

Definition 6.1. Assume φ is the time one map of a discrete random dynamical system, $S(\omega)$ is a random isolated invariant set for φ and $P = (N(\omega), L(\omega))$ is a random filtration pair for $S(\omega)$. Denote $h_P(S, \varphi)$ the random homotopy class $[\varphi_P]$ on the random pointed space $N_L(\omega)$ with φ_P a representative element, then we define the *random Conley index $h(S, \varphi)$ for $S(\omega)$* to be the random shift equivalent class of $h_P(S, \varphi)$.

Remark 6.1. It is well known that the deterministic Conley index is defined for autonomous flows or maps, but for discrete RDS we confront, the random homeomorphism is non-autonomous. As for Conley index for non-autonomous flows, some authors have dealt with them, see [22, 23, 24, 14] among others. They treat them as skew-product flows by the method of extending phase space. But their method is inappropriate for our purpose in spite that a discrete RDS

can be seen as a measurable discrete skew-product flow. In fact, their method depends crucially on the fact that the hull of vector field function $H(f)$ is compact, see [22, 23, 24, 14] for details. But for RDS, the probability space (Ω, \mathcal{F}) is only a measurable space and we can not in general assume that it is a compact topology space, which is too restrictive for applications.

The following two theorems state that the random Conley index for discrete RDS has the similar properties to that of Conley index for deterministic maps.

Theorem 6.1. (Continuation property) *Assume $\varphi_\lambda, \lambda \in [0, 1]$ is a family of random homeomorphisms, which depends continuously on λ . If $N(\omega)$ is a random isolating neighborhood for each $\varphi_\lambda, \lambda \in [0, 1]$, then the random Conley index $h(S_\lambda, \varphi_\lambda)$ for φ_λ is independent of $\lambda \in [0, 1]$, i.e. $h(S_\lambda, \varphi_\lambda) = h(S_0, \varphi_0)$, where $S_\lambda(\omega) := \text{Inv}(N(\omega), \varphi_\lambda)$, $\lambda \in [0, 1]$ stands for the random isolated invariant set for φ_λ in $N(\omega)$.*

Proof. The proof follows immediately from Theorem 4.1. \square

We use $\underline{0}$ to denote the random Conley index of random pointed spaces consisting of just one random point with random constant maps as their corresponding random pointed space maps. Given a random isolated invariant set $S(\omega)$ in a random pointed space $N_L(\omega)$ with φ_P the corresponding random pointed space map, by the definition of random shift equivalence it is easy to verify that the random Conley index $h(S, \varphi)$ for $S(\omega)$ is $\underline{0}$ if and only if $\varphi_P \simeq f$ and $\varphi_P^n \simeq f^n$ for some random map $f(\omega, \cdot) : N_L(\omega) \rightarrow N_L(\theta\omega)$ and some measurable $n = n(\omega)$, where $f^n : N_L(\omega) \rightarrow N_L(\theta_n\omega)$ is the base-point valued random constant map.

The random Conley index can be used to study the structure of random invariant set, see the following theorem.

Theorem 6.2. (Ważewski property) *Assume $S(\omega)$ is a random isolated invariant set and the random Conley index for $S(\omega)$ is not trivial, i.e. $h(S, \varphi) \neq \underline{0}$, then $S(\omega) \neq \emptyset$ almost surely when θ is ergodic under \mathbb{P} .*

Proof. If $S(\omega) = \emptyset$ with positive probability, then $S(\omega) = \emptyset$ almost surely by the ergodicity of θ and the invariance of $S(\omega)$. Hence we have (\emptyset, \emptyset) is a random filtration pair for $S(\omega)$. It follows that $h(S, \varphi) = \underline{0}$, a contradiction. \square

7 Relations between time-continuous RDS and discrete one

Assume ϕ is a time-continuous RDS and consider its time- h map— $\varphi_h(\omega) := \phi(h, \omega) : X \rightarrow X$, where $h > 0$. Then $\varphi_h^k(\omega) = \phi(kh, \omega)$ and φ_h is a discrete RDS (recalling that throughout the paper we identify a random homeomorphism with the discrete RDS generated by it) generated by the time- h map of the time-continuous RDS ϕ . For time-continuous RDS, replacing $n \in \mathbb{Z}$ by $t \in \mathbb{R}$, we can introduce isolated invariant set and isolating neighborhood similar to Definition 3.2.

In this section we simply discuss the relation of isolated invariant sets between time-continuous RDS and the discrete one generated by its time- h map.

Theorem 7.1. *Assume ϕ is a time-continuous RDS and there exists a $\delta > 0$ such that for $\forall h \in (0, \delta]$, $S(\omega)$ is an isolated invariant set of φ_h , then $S(\omega)$ is an isolated invariant set of ϕ .*

Proof. It is obvious that $S(\omega)$ is invariant for ϕ by the fact that $S(\omega)$ is invariant with respect to φ_h for $\forall h \in (0, \delta]$, so we only need to show it is isolated with respect to ϕ . Assume $N(\omega)$ is an isolating neighborhood of $S(\omega)$ with respect to φ_h for some $h \in (0, \delta]$. By the fact

$$S(\omega) \subset \text{Inv}(N(\omega), \phi) \subset \text{Inv}(N(\omega), \varphi_h) = S(\omega)$$

we immediately obtain the desired result. \square

Theorem 7.2. *Assume ϕ is a time-continuous RDS and $S(\omega)$ is an isolated invariant set of ϕ . Moreover, if there exists an isolating neighborhood $N(\omega)$ of $S(\omega)$ such that the map*

$$t \rightarrow N(\theta_t \omega) \in \mathcal{K}(X) \quad (17)$$

is continuous (in general by the definition of random compact set the map is only measurable), where $\mathcal{K}(X)$ denotes the space of non-empty compact subsets of X endowed with the Hausdorff metric. Then for $\forall h > 0$, $S(\omega)$ is an isolated invariant set of φ_h .

Proof. The idea of proof is originated from [13]. For fixed $h > 0$, by the fact $S(\omega)$ is isolated (with respect to ϕ) by $N(\omega)$, the compactness of $[0, h]$ and the continuity of the map (17) we can choose a compact neighborhood $\tilde{N}(\omega) \subset N(\omega)$ of $S(\omega)$ such that

$$\phi(t, \omega) \tilde{N}(\omega) \subset N(\theta_t \omega), \quad \forall t \in [0, h].$$

In fact we can choose

$$\tilde{N}(\omega) = \bigcap_{t \in [0, h]} \phi(-t, \theta_t \omega) N(\theta_t \omega) = \bigcap_{t \in [0, h] \cap \mathbb{Q}} \phi(-t, \theta_t \omega) N(\theta_t \omega),$$

where the second “=” holds by the continuity of the map (17). Clearly $\tilde{N}(\omega)$ obtained in this way is a random compact set. We only need to show that $S(\omega)$ is the maximal invariant random compact set in $\tilde{N}(\omega)$ with respect to φ_h , i.e.

$$S(\omega) = \text{Inv}(\tilde{N}(\omega), \varphi_h).$$

For $\forall x \in \text{Inv}(\tilde{N}(\omega), \varphi_h)$, we have

$$\varphi_h^k(\omega, x) \in \tilde{N}(\theta_{kh} \omega), \quad \forall k \in \mathbb{Z}.$$

By the choice of $\tilde{N}(\omega)$ we easily obtain that

$$\phi(t, \omega, x) \in N(\theta_t \omega), \quad \forall t \in \mathbb{R},$$

which implies that $x \in S(\omega)$. Hence we obtained the desired result. \square

Remark 7.1. Of course assuming the map (17) being continuous is very restrictive for applications, but there is some cases the assumption does hold. For instance, when an RDS admits a random attractor (in the sense of [19]) then there is compact neighborhood of the attractor satisfying this assumption which play the role of isolating neighborhood, noting that for any invariant random compact set the continuity of the map (17) naturally holds.

8 Several simple examples

In this section we give several simple examples to illustrate our results.

Example 8.1. Assume the 2×2 random matrix $A(\omega) = \begin{pmatrix} a(\omega) & 0 \\ 0 & b(\omega) \end{pmatrix}$ is

the time-one map of a discrete random dynamical system with phase space \mathbb{R}^2 and assume $N(\omega) \equiv \{(x, y) \mid x^2 + y^2 \leq 1\}$.

(i) If $0 < a(\omega) < 1, 0 < b(\omega) < 1$ for $\forall \omega \in \Omega$, then it is easy to see (in fact, we have $A(\omega)N(\omega) \subset \text{int}N(\theta\omega)$) that $N(\omega)$ is a random isolating block and $S(\omega) \equiv \{0\}$ is the corresponding random isolated invariant set. It is obvious that $N^-(\omega) \equiv \emptyset$ and $P = (N(\omega), \emptyset)$ is a random filtration pair for $S(\omega)$. It is easy to see that any power of the random pointed space map A_P on the random pointed space $N_L(\omega) = N(\omega) \cup [\emptyset]$ is not random homotopic to the base-point valued random constant map, hence we have $h(S, A) \neq \underline{0}$.

(ii) If $a(\omega) > 1, b(\omega) > 1$ for $\forall \omega \in \Omega$, then $N(\omega)$ is a random isolating block and $S(\omega) \equiv \{0\}$ is the corresponding random isolated invariant set. Assume $P = (N(\omega), L(\omega))$ is a random filtration pair for $S(\omega)$, where $L(\omega)$ is a sufficiently small neighborhood of $N^-(\omega)$ (it is obvious that $N^-(\omega) \neq \emptyset$ for all $\omega \in \Omega$). The random pointed space $N_L(\omega)$ can be regarded as S^2 for each ω and it is easy to see that $A_P(\omega) : N_L(\omega) \rightarrow N_L(\theta\omega)$ is a surjection. Hence any power of A_P is not random homotopic to the base-point valued random constant map, i.e. $h(S, A) \neq \underline{0}$.

(iii) If $a(\omega) > 1, 0 < b(\omega) < 1$ for $\forall \omega \in \Omega$ and assume $N(\omega)$ is a random isolating block for $S(\omega) \equiv \{0\}$ (it is easy to see that there is possibility that $N(\omega)$ is not a random isolating block for $S(\omega)$, so we make such assumption). It is easy to see that $N^-(\omega) \neq \emptyset$ for each ω and assume $L(\omega)$ is a sufficiently small neighborhood of $N^-(\omega)$, then $P = (N(\omega), L(\omega))$ is a random filtration pair for $S(\omega)$. Then by a simple verification we can obtain that $h(S, A) \neq \underline{0}$.

Example 8.2. Consider the logistic model:

$$\dot{x} = rx(1 - \frac{x}{K}),$$

where $0 < r \leq 1, K > 0$ are constants. By Euler approximation method we can obtain the corresponding logistic difference equation as follows:

$$x_{n+1} = x_n + hr x_n(1 - \frac{x_n}{K}),$$

where h is the step size of Euler method. Now assume that r is perturbed by a real noise, say,

$$r(\omega) = r + \xi(\omega)$$

with $|\xi(\omega)| < r$. We now consider some properties of the perturbed difference equations:

$$x_{n+1} = x_n + hr(\theta_{nh}\omega)x_n(1 - \frac{x_n}{K}).$$

To this end, consider the following parameterized difference equations:

$$x_{n+1} = x_n + hr_\lambda(\theta_{nh}\omega)x_n(1 - \frac{x_n}{K}) \quad (18)$$

with $r_\lambda(\omega) = r + \lambda\xi(\omega)$, $\lambda \in [0, 1]$. Then for each $\lambda \in [0, 1]$, (18) generates a discrete RDS, denoting it by φ_λ . For the generation of discrete RDS from random difference equation, the reader can refer to [1] for details. Set $X = [0, +\infty)$.

(1) Let $N = [0, M]$ with $M > K$. Since $r_\lambda(\omega) > 0$ for all ω , we have

$$\varphi_\lambda(h, \omega)N \subset \text{int}N$$

for small h , which implies that N is an isolating neighborhood for the family φ_λ , $\lambda \in [0, 1]$. It is clear that when $\lambda = 0$, i.e. for the unperturbed deterministic difference equation, the Conley index for the isolated invariant set $S_0 = [0, K]$ is nontrivial. Hence by the continuity property of random conley index we have that $h(S_\lambda, \varphi_\lambda) = h(S_0, \varphi_0) \neq \underline{0}$, $\lambda \in [0, 1]$. Therefore, by the Waz wski property of random Conley index we obtain that there exists an invariant random compact set in N for the perturbed difference equation.

(2) Consider any closed interval \tilde{N} containing K but not containing 0. Then it is easy to verify that

$$\varphi_\lambda(h, \omega)\tilde{N} \subset \text{int}\tilde{N}$$

when h is small, which implies that \tilde{N} is an isolating neighborhood for the family φ_λ , $\lambda \in [0, 1]$. Therefore, by the similar argument to that of (1) we obtain that there exists an invariant random compact set in \tilde{N} for the perturbed difference equation for small h . Since the interval \tilde{N} can be chosen sufficiently small, we can conclude that $\{K\}$ is an invariant random compact set for the perturbed difference equation, which coincides with our intuition.

(3) Consider any closed interval like $[0, a]$ with $a < K$. Denote $\hat{N} = [0, a]$, then it is easy to see that

$$[0, K) \supset \varphi_\lambda(h, \omega)\text{int}\hat{N} \supset \hat{N}$$

for small h , which implies that \hat{N} is an isolating neighborhood for the family φ_λ , $\lambda \in [0, 1]$. Again by the same argument to that of (1), (2) we obtain that $\{0\}$ is an invariant random compact set for the perturbed difference equation for small h , which coincides also with our intuition.

Example 8.3. Consider the Lorenz system in \mathbb{R}^3 described by the equations:

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = xy - \beta z \end{cases}$$

with $\sigma, \rho, \beta > 0$. By Euler approximation method we obtain the corresponding discrete Lorenz system:

$$X_{n+1} = (hB + I)X_n + hF(X_n), \quad (19)$$

where h is the step size of Euler approximation method, $X_n = (x_n, y_n, z_n)^\top$, I is the identity matrix, $B = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$, and $F(X_n) = (0, -x_n z_n, x_n y_n)^\top$.

Assume that σ, ρ, β is perturbed by real noises, say,

$$\begin{aligned} \sigma(\omega) &= \sigma + \xi(\omega), \\ \rho(\omega) &= \rho + \eta(\omega), \\ \beta(\omega) &= \beta + \zeta(\omega). \end{aligned}$$

Denote $\sigma_\lambda(\omega) = \sigma + \lambda\xi(\omega)$, $\rho_\lambda(\omega) = \rho + \lambda\eta(\omega)$, $\beta_\lambda(\omega) = \beta + \lambda\zeta(\omega)$, $\lambda \in [0, 1]$. Consider the family of parameterized difference systems:

$$X_{n+1} = (hB_\lambda(\theta_{nh}\omega) + I)X_n + hF(X_n) \quad (20)$$

with

$$B_\lambda(\theta_{nh}\omega) = \begin{pmatrix} -\sigma_\lambda(\theta_{nh}\omega) & \sigma_\lambda(\theta_{nh}\omega) & 0 \\ \rho_\lambda(\theta_{nh}\omega) & -1 & 0 \\ 0 & 0 & -\beta_\lambda(\theta_{nh}\omega) \end{pmatrix}.$$

When $\lambda = 0$, (20) corresponds to the unperturbed discretized Lorenz system by Euler approximation method; when $\lambda = 1$, (20) corresponds to the discrete Lorenz system perturbed by real noise. Similar to Example 8.2, (20) generates a discrete RDS for each λ and denote it by φ_λ .

(1) Assume that for each $\lambda \in [0, 1]$, $0 < \rho_\lambda(\omega) < \sigma_\lambda(\omega) \leq 1$ almost surely. Noting that

$$\begin{aligned} \langle X_{n+1}, X_n \rangle &= \langle (hB_\lambda(\theta_{nh}\omega) + I)X_n, X_n \rangle + h\langle F(X_n), X_n \rangle \\ &= \langle (hB_\lambda(\theta_{nh}\omega) + I)X_n, X_n \rangle \\ &= (1 - h\sigma_\lambda(\theta_{nh}\omega))x_n^2 + h(\sigma_\lambda(\theta_{nh}\omega) + \rho_\lambda(\theta_{nh}\omega))x_n y_n \\ &\quad + (1 - h)y_n^2 + (1 - h\beta_\lambda(\theta_{nh}\omega))z_n^2, \end{aligned}$$

we have

$$\begin{aligned} |\langle X_{n+1}, X_n \rangle| &\leq (1 - h\sigma_\lambda(\theta_n\omega))x_n^2 + (1 - h)y_n^2 + (1 - h\beta_\lambda(\theta_n\omega))z_n^2 \\ &\quad + \frac{h}{2}(\sigma_\lambda(\theta_n\omega) + \rho_\lambda(\theta_n\omega))(x_n^2 + y_n^2) \\ &\leq (1 - \frac{h(\sigma_\lambda(\theta_n\omega) - \rho_\lambda(\theta_n\omega))}{2})x_n^2 + (1 - h\beta_\lambda(\theta_n\omega))z_n^2 \\ &\quad + [1 - \frac{h}{2}(2 - \sigma_\lambda(\theta_n\omega) - \rho_\lambda(\theta_n\omega))]y_n^2 \\ &< \|X_n\|^2 \end{aligned}$$

for small h by the assumption $0 < \rho_\lambda(\omega) < \sigma_\lambda(\omega) \leq 1$, where the last “ $<$ ” holds if $X_n \neq (0, 0, 0)$. That is, we have shown that $\|X_{n+1}\| \leq \|X_n\|$, which implies that any ball $B_r(0)$ centered at the origin with radius $r > 0$ is an isolating neighborhood for φ_λ , $\lambda \in [0, 1]$. It is easy to verify that $\{(0, 0, 0)\}$ is an isolated invariant set for the unperturbed discrete Lorenz system (19) with nontrivial Conley index. Hence by the continuity property of random Conley index we have that $h(S_\lambda, \varphi_\lambda) = h(S_0, \varphi_0) \neq \mathbb{Q}$, $\lambda \in [0, 1]$. Therefore, by the Waz wski property of random Conley index we obtain that there exists an invariant random compact set in $B_r(0)$ for the perturbed discrete Lorenz system. Since the above argument holds for arbitrary $r > 0$, we obtain that $\{(0, 0, 0)\}$ is an invariant random compact set for the perturbed discrete Lorenz system.

(2) Assume that for each $\lambda \in [0, 1]$, $\sigma_\lambda(\omega), \beta_\lambda(\omega) > 0$ almost surely. We consider the Lorenz system on the sub-manifold: $M = \{(x, y, z) \in \mathbb{R}^3 | y = 0\}$ (clearly M is an invariant random set of RDS φ_λ , for each $\lambda \in [0, 1]$), then by the computation in (1) we have

$$\langle X_{n+1}, X_n \rangle = (1 - h\sigma_\lambda(\theta_n\omega))x_n^2 + (1 - h\beta_\lambda(\theta_n\omega))z_n^2.$$

Hence for small h , we have

$$|\langle X_{n+1}, X_n \rangle| \leq \|X_n\|^2,$$

i.e.

$$\|X_{n+1}\| < \|X_n\|$$

almost surely whenever $X_n \neq (0, 0, 0)$. This indicates that when restricted on the sub-manifold M , φ_λ , $\lambda \in [0, 1]$ admits any neighborhood of $(0, 0, 0)$ in M as an isolating neighborhood. Clearly the Conley index for φ_0 , i.e. the unperturbed discrete Lorenz system (19), is nontrivial. Similar to the argument of (1) we immediately obtain that $\{(0, 0, 0)\}$ is an invariant random compact set for the perturbed discrete Lorenz system when it is restricted on M .

Example 8.4. Assume that φ is a locally tempered analytic random diffeomorphism in \mathbb{C}^d with a fixed point $x = 0$ and its linearization $A(\omega)$ satisfies the conditions of the multiplicative ergodic theorem. Moreover, if all Lyapunov exponents $\lambda_i(\omega)$, $1 \leq i \leq p(\omega)$, have the same sign and are non-resonant, then by (iii) of Main Theorem of [9] we have φ is analytically conjugate to its linear part. That is, if we write $\varphi(\omega, x)$ as

$$\varphi(\omega, x) = A(\omega)x + f(\omega, x),$$

where $A(\omega) = D\varphi(\omega, 0) \in Gl(d, \mathbb{C})$, $f(\omega, 0) = 0$ and $Df(\omega, 0) = 0$, then there exists an analytic random diffeomorphism $h(\omega, x)$ defined in a tempered ball $V(\omega)$ with $h(\omega, 0) = 0$ such that

$$h(\theta\omega, \varphi(\omega, x)) = A(\omega)h(\omega, x), \text{ for } x \in V(\omega) \text{ a.e.} \quad (21)$$

For the details, the reader can refer to [9]. Since all Lyapunov exponents $\lambda_i(\omega)$ have the same sign, it is clear that $\{0\}$ is an isolated invariant set for $A(\omega)x$ and

$h(\{0\}, A(\omega)) \neq \underline{0}$. By (21) we obtain that $\{0\}$ is an isolated invariant set for φ . Further more, the random Conley index of φ is the same as that of linear part. To see this, consider the family of random diffeomorphisms:

$$\varphi_\lambda(\omega, x) = A(\omega)x + \lambda f(\omega, x), \quad \lambda \in [0, 1].$$

By (iii) of Main Theorem of [9] we have for each $\lambda \in [0, 1]$, there exists $V_\lambda(\omega)$ such that φ_λ is analytically conjugate to $A(\omega)x$ in $V_\lambda(\omega)$ and $V_\lambda(\omega)$ play the role of isolating neighborhood with respect to φ_λ , isolating $\{0\}$. By a compactness argument there exists a $V(\omega)$ such that φ_λ is analytically conjugate to $A(\omega)x$ in $V(\omega)$ for each $\lambda \in [0, 1]$ and $V(\omega)$ is an isolating neighborhood for each φ_λ . Hence by the continuity property of random Conley index we have $h(S_\lambda, \varphi_\lambda) = h(\{0\}, \varphi_0)$, where $S_\lambda(\omega) := \text{Inv}(V(\omega), \varphi_\lambda)$, $\lambda \in [0, 1]$. Since $V(\omega)$ can be chosen sufficiently small such that $S_1(\omega) = \{0\}$, we obtain that $h(\{0\}, \varphi_1) = h(\{0\}, \varphi_0)$, i.e. $h(\{0\}, \varphi) = h(\{0\}, A(\omega))$.

If φ is a C^∞ locally tempered random diffeomorphism in \mathbb{R}^d with a hyperbolic fixed point $x = 0$ and its linearization $A(\omega)$ satisfies the conditions of the multiplicative ergodic theorem. And all Lyapunov exponents satisfy non-resonance condition, then by (ii) of Theorem 1.1 of [10] we have φ is conjugate to its linear part. By the similar argument to that of analytic case in \mathbb{C}^d we can obtain that $\{0\}$ is an isolated invariant set for φ and it has the same random Conley index with respect to φ as it has with respect to its linear part.

Acknowledgements

I am most indebted to my advisor, Professor Yong Li, not only for his direct helpful suggestions but primarily for his continual instruction, encouragement and support over all these years. I am very grateful to Professor Yingfei Yi for helpful discussions and valuable suggestions during the period when he visit Jilin University, to Professor Ludwig Arnold for his kind encouragement and to Professor Dave Richeson for his patient explaining his joint paper with Professor John Franks.

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